Int. J. Heat Mass Transfer. Vol. 8, pp. 1047–1050. Pergamon Press 1965. Printed in Great Britain

SHORTER COMMUNICATION

THE RADIAL VARIATION OF THE EDDY VISCOSITY IN COMPRESSIBLE TURBULENT JET FLOWS

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(*Received* 6 *November* 1964 *and in revised form* 2 *March* 1965)

NOMENCLATURE

- C_{p_i}, specific heat at constant pressure of species i at reference temperature;
- H, stagnation enthalpy;
- p, static pressure;
- R_0, universal gas constant;
- r, radial co-ordinate;
- T, static temperature;
- u, v, velocity components;
- W, molecular weight of mixture;
- x, streamwise co-ordinate;
- Y_1, mass fraction of oxygen;
- Y_2, mass fraction of hydrogen;
- Y_3, mass fraction of nitrogen;
- Δ_i, static enthalpy of species i at reference temperature;
- ϵ, compressible kinematic eddy viscosity coefficient;
- $\bar{\epsilon}$, incompressible kinematic eddy viscosity coefficient;
- μ, dynamic viscosity coefficient;
- ρ, density.

Subscripts

- $\mathbb{C}$, centerline;
- j, jet;
- s, stagnation conditions;
- e, external stream;
- t, turbulent.

INTRODUCTION

ALTHOUGH our knowledge of the actual transport mechanism of turbulent mixing processes is still rather limited, recent experimental and theoretical work has provided more insight into the problem and helped clarify several points which were open to question until now. For example, it could be shown that Prandtl's expression for the incompressible turbulent kinematic viscosity coefficient $\bar{\epsilon}$ is incorrect for a jet exhausting into a stream with uniform velocity [1]. A comparison of theoretical predictions with experimental data [1] indicated errors of more than 100 per cent when the aforementioned relation for $\bar{\epsilon}$ was used. However, good agreement between theory and experiments was obtained when $\bar{\epsilon}$ was taken proportional to the product of the half boundary and the centerline velocity $u_{\mathbb{C}}$ instead of the velocity difference $\Delta u_{\max}$ as introduced by Prandtl.

Reference 2 suggested a modification of Prandtl's expression for compressible flows. Instead of relating the kinematic viscosity coefficient ϵ to the velocity difference, it was proposed to relate the dynamic coefficient $\mu_t = \rho\epsilon$ to the maximum mass flux difference. This model yields good results for jets exhausting into a quiescent atmosphere, however, it fails for jets exhausting into a stream with uniform velocity. Similarly as for incompressible jets, theory and experiments agree quite favorably when the mass flux difference is replaced by the mass flux at the centerline [1]. The formulation of [2] is discussed in detail in [3] and [4]. In [4] it was shown that for a jet exhausting into a quiescent atmosphere the same result can be obtained by linearizing the momentum equation.

An expression which relates the incompressible and the compressible kinematic viscosity coefficient directly has already been given by Ting and Libby in 1960 [5]. The expression they proposed includes the radial variation of ϵ caused by the density gradients in the mixing region. A comparison to other formulations or experimental data has not yet been made.

It is the purpose of the present analysis to compare the aforementioned transformation with relations developed in [1]. In that reference the streamwise momentum equation is solved for ϵ such that if all flow quantities appearing in the expression for ϵ are determined experimentally ϵ and μ_t can be calculated. Data so obtained may serve to confirm the validity of already existing formulations for turbulent transport coefficients.

ANALYSIS

In [5] Ting and Libby showed that the turbulent compressible viscosity coefficient ϵ can be related to its incompressible value. Employing the Mager transformation [6] and assuming that the moment about the axis of turbulent shear stress over an infinitesimal mass is preserved, the authors derived the following relation for the case of axisymmetric jets

$$\frac{\epsilon}{\bar{\epsilon}} = \frac{2\rho_0}{(r\rho)^2}\int_0^r \rho r' \,\mathrm{d}r', \tag{1}$$

where ρ_0 denotes a reference density. If ρ_0 is chosen to be equal to $\rho_{\mathfrak{L}}$, then the centerline values of ϵ and $\bar{\epsilon}$ are formally the same since

$$\lim_{r\to 0} \frac{\epsilon_{\mathfrak{L}}}{\bar{\epsilon}} = 1 \tag{2}$$

If now $\bar{\epsilon}$ is specified, the compressible value of ϵ can be calculated provided the density profiles in the mixing region are known.

A different approach is taken in [1]; there the momentum equation is solved for ϵ.

$$\epsilon = \left[\int_0^r \frac{\partial}{\partial x}(\rho u^2)\, r'\,\mathrm{d}r' + \rho\, uvr\right]\left[\rho r \frac{\partial u}{\partial r}\right]^{-1} \tag{3}$$

Equation (3) is considerably simplified as r approaches zero. Then ϵ reduces to

$$\lim_{r\to 0} \epsilon_{\mathfrak{L}} = \epsilon = \left(\frac{u\dfrac{\partial u}{\partial x}}{\dfrac{2\,\partial^2 u}{\partial r^2}}\right)_{\mathfrak{L}} \tag{4}$$

Values of ϵ and $\epsilon/\epsilon_{\mathfrak{L}}$ may now be evaluated from equations (3) and (4) provided the density and velocity profiles and their derivatives with respect to the streamwise and radial co-ordinate can be determined. If this information is available, results of equations (3) and (4) may serve to prove the validity of equation (1).

The experimental determination of derivatives is usually very difficult. It is, therefore, desirable to correlate the measurements first through suitable expressions which then allow determination of the derivatives analytically. Several expressions have been suggested for the representation of the radial profiles for jet flows. In the present analysis Forstall's and Shapiro's cosine profile [7] will be used. It relates the concentration of the injected gas and the velocity difference to the ratio of the radial co-ordinate r and the half boundary $r_{\frac{1}{2}}$ as follows:

$$\frac{Y_2}{Y_{2\mathfrak{L}}} = \frac{u_e - u}{u_e - u_{\mathfrak{L}}} = \tfrac{1}{2}\left[1 + \cos\left(\frac{\pi}{2}\frac{r}{r_{\frac{1}{2}}}\right)\right] \tag{5}$$

Though the asymptotic behavior is not described accurately by equation (5), it correlates the experimental data with good accuracy and provides a convenient means for evaluating the velocity and concentration derivatives. An example for a correlation of hydrogen concentration profiles of [1] is given in Fig. 1. In addition, the spreading of the jet (i.e. the half boundary $r_{\frac{1}{2}}$), and the centerline values of concentration and velocity must be specified. The results of [1] indicate that these quantities may be approximated by equations of the form

$$Y_{2\mathfrak{L}} = a_1 x^{-n_1} \qquad r_{\frac{1}{2}} = b_1 x^{m_1} \tag{6}$$

and

$$\frac{u_e - u_{\mathfrak{L}}}{u_e - u_j} = a_2 x^{-n_2} \qquad r_{\frac{1}{2}} = b_2 x^{m_1}, \tag{7}$$

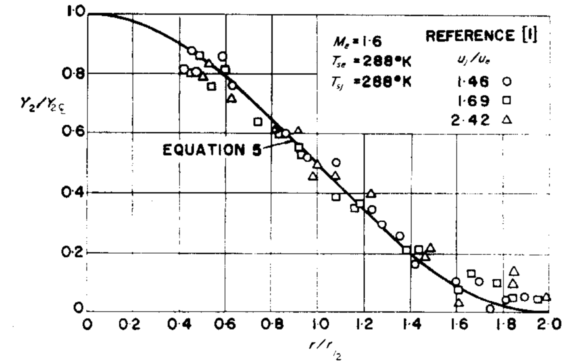

FIG. 1. Correlation of normalized mass concentration profiles for hydrogen air mixing in a compressible turbulent jet flow.

where a_1, a_2, b_1, b_2, m_1, m_2, n_1, and n_2 are experimental constants.

The density derivatives in equation (3) are obtained by differentiating the equation of state

$$\frac{\partial \rho}{\partial x} = \frac{p}{R_0 T}\left(\frac{\partial W}{\partial x} - \frac{W \partial T}{T \partial x}\right). \tag{8}$$

The derivative of the molecular weight is readily available since it depends on the mass fraction of the injected gas only and can thus be evaluated by means of equations (5) and (6). The only unknown quantities in equation (8) are the temperature T and its derivative $(\partial T/\partial x)$. If the stagnation temperature is measured, both can be computed from the stagnation enthalpy in the mixing region. Then

$$T = \left\{\sum_{i=1}^{i=3} Y_i\,[C_{pi}(T_s - T_r)] - \frac{u^2}{2}\right\}\left[\sum_{i=1}^{i=3} Y_i C_{pi}\right]^{-1} + T_r \tag{9}$$

In equation (9) T is linearly related to the enthalpy and is referred to a suitable chosen reference temperature T_r. When the Le_t, Pr_t, and the Sc_t numbers are assumed to be equal to unity, the Crocco integral can be employed to determine the temperature:

$$T = \{[H_e(u - u_j) + H_j(u_e - u)](u_e - u_j)^{-1} - \tfrac{1}{2}u^2 - \sum_{i=1}^{i=3} Y_i \Delta_i\}\,[\sum_{i=1}^{i=3} Y_i C_{pi}]^{-1} + T_r \tag{10}$$

Now ϵ and $\epsilon/\epsilon_{\mathfrak{L}}$ may be found by numerical integration of equations (1) and (3), where the transverse velocity component v in equation (3) is evaluated from integration of the continuity equation

$$v = -\frac{1}{\rho r}\int_0^r \frac{\partial}{\partial x}(\rho u) r'\,\mathrm{d}r' \tag{11}$$

RESULTS

As mentioned in the previous section, several experimental constants are needed in order to determine $\epsilon/\epsilon_{\mathfrak{L}}$

and $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$. These constants are taken from [1] and, therefore, the present calculation is carried out for the test conditions stated therein. The constants n_1 and m_1 in equation (6) were found to be $n_1 = 2$, and $m_1 = 0{\cdot}65$. Since the Pr_t, Sc_t, and Le_t numbers were close to unity for several test series, the Crocco integral is used here. Then momentum and mass diffusion become identical so that the spreading rates are the same; hence $n_2 = n_1$ and $m_2 = m_1$. The constants a and b do not have to be specified since they cancel out, when ϵ is referred to its centerline value.

The test conditions of [1] are as follows. Two jets are arranged coaxially and an injection gas is discharged through a subsonic or supersonic nozzle at the center into an airstream moving at a constant supersonic speed. The Mach number of the airstream is $M_e = 1{\cdot}6$, the stagnation temperature T_{se} and T_{sj} are 288°K. The static pressure in the mixing region is 1 atm. Three different injection gases, hydrogen, helium and argon, are used in the various test series.

For the present calculation only hydrogen is considered, since it provides the largest density variation in the mixing region and should, therefore, have a larger effect then helium and argon on the radial variation of ϵ and $\rho\epsilon$. The initial velocity ratios are chosen as $u_j/u_e = 0{\cdot}1$, $0{\cdot}5$, $0{\cdot}9$, $2{\cdot}0$ and $4{\cdot}0$ and the radial profiles of $\epsilon/\epsilon_{\mathbb{C}}$ and $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$ are calculated for three different x-stations: one, relatively close to the jet, where $x/x_0 = 1{\cdot}5$, the second somewhat further downstream ($x/x_0 = 5{\cdot}0$), and the third where $x/x_0 = 10{\cdot}0$. Here x_0 denotes the length of the potential core, i.e. the distance from the jet discharge to the point where the centerline values of concentration and velocity start to change.

The results obtained from equation (1), (3) and (4) are shown in Figs. 2 through 4. In Fig. 2 the radial profiles of

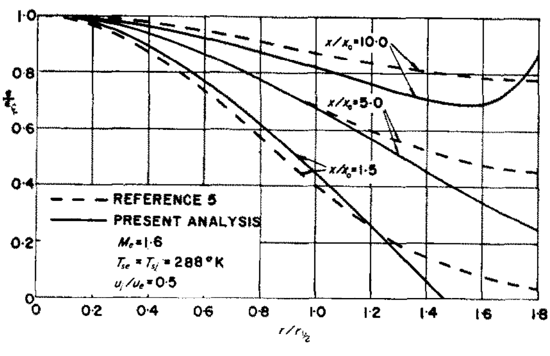

FIG. 2. Radial variation of the nondimensionalized eddy viscosity coefficient $\epsilon/\epsilon_{\mathbb{C}}$ in a hydrogen air mixture.

$\epsilon/\epsilon_{\mathbb{C}}$ are plotted vs $r/r_{\frac{1}{2}}$ for the velocity ratio $u_j/u_e = 0{\cdot}5$. The largest value of $\epsilon/\epsilon_{\mathbb{C}}$ is found at the centerline and with increasing r, $\epsilon/\epsilon_{\mathbb{C}}$ deviates from one quite markedly. In the range $0 \leq r/r_{\frac{1}{2}} \leq 1{\cdot}2$ the results of the present analysis agree quite favorably with those of [5]. For larger values of $r/r_{\frac{1}{2}}$ the present analysis becomes less accurate and the curves approach infinity at $r/r_{\frac{1}{2}} = 2$. Clearly, the limiting value ($r \to \infty$) should at least be finite if not zero. This discrepancy is explained by the incorrect asymptotic behavior of the velocity and concentration profiles as given by equation (5).

In Fig. 3 the radial variation of $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$ is presented. It is seen that the radial dependence of $\rho\epsilon$ on $r/r_{\frac{1}{2}}$ is not

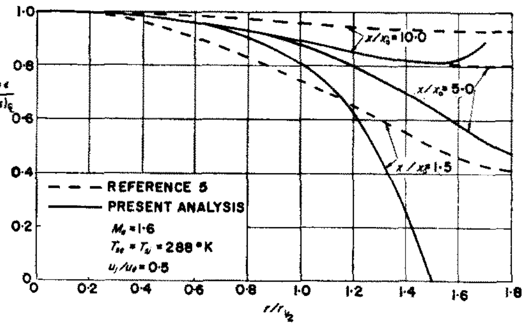

FIG. 3. Radial variation of the nondimensionalized turbulent viscosity coefficient $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$ in a hydrogen air mixture.

as pronounced as that of $\epsilon/\epsilon_{\mathbb{C}}$. Up to values of $r/r_{\frac{1}{2}} = 1{\cdot}2$, the agreement with [5] is again quite satisfactory and within 10 per cent.

The dependence of $\rho_{\mathbb{C}}$ on the ratio of the initial velocities u_j/u_e is shown in Fig. 4, where the values of $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$ at the half boundary are plotted versus u_j/u_e. It

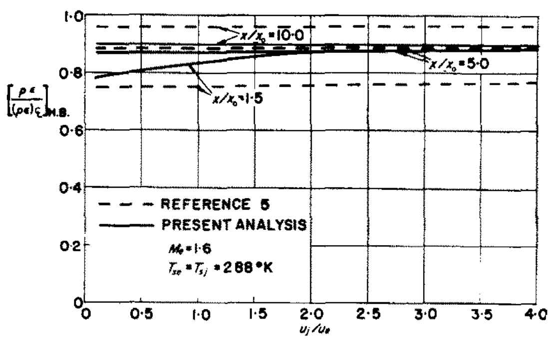

FIG. 4. The nondimensionalized turbulent viscosity coefficient $\rho\epsilon/(\rho\epsilon)_{\mathbb{C}}$ at the half boundary as a function of the velocity ratio u_j/u_e. Hydrogen injection.

is interesting to note that $\rho\epsilon$ is almost independent of u_j/u_e for the entire range of velocity ratios investigated.

CONCLUSIONS

The radial variation of the turbulent kinematic and dynamic viscosity coefficients in the mixing region of two coaxial heterogeneous compressible jets was investigated. Based on experimental results of [1], radial profiles for ϵ and $\rho\epsilon$ were obtained for various initial velocity ratios of the two streams.

It was found that the kinematic viscosity coefficient ϵ varies appreciably in the radial direction, whereas the dynamic coefficient $\rho\epsilon$ does not exhibit such a strong radial dependence. This result confirms that the assumption of $\rho\epsilon$ remaining constant in the radial direction and varying in the streamwise direction as discussed in [2, 3, and 4] is a reasonable approximation to its actual behavior. It was also shown that the viscosity coefficients ϵ and $\rho\epsilon$ do not strongly depend on the initial velocity ratio of the two streams.

The results of the present analysis were compared with those obtained from the transformation for the compressible kinematic viscosity coefficient of [5]. It is indicated that the agreement is quite satisfactory. Therefore, it can be concluded that this transformation describes the radial variation of ϵ with good accuracy, provided $\bar{\epsilon}$ is specified correctly.

ACKNOWLEDGEMENT

The authors take this opportunity to thank Dr. Antonio Ferri for many helpful discussions of the work presented here.

REFERENCES

1. V. Zakkay, E. Krause and S. D. K. Woo, *Turbulent Transport Properties for Axisymmetric Heterogeneous Mixing*. Paper presented at the AIAA Meeting, New New York, Jan. 1964. Also Polytechnic Institute of Brooklyn, PIBAL Rept. 813. Also *AIAA J.* **2**, 1939–1947 (1964).
2. A. Ferri, P. A. Libby and V. Zakkay, Theoretical and experimental investigation of supersonic combustion, in *High Temperatures in Aeromautics*, pp. 55–118. Pergamon Press, Oxford (1962).
3. L. J. Alpinieri, *An Experimental Investigation of the Turbulent Mixing of Non-homogeneous Coaxial Jets*, Polytechnic Institute of Brooklyn, PIBAL Rept. 789 (August 1963); also Turbulent mixing of coaxial jets, *AIAA J.* **2**, 1560–1568 (1964).
4. G. Kleinstein, Mixing in turbulent, axially symmetric free jets, *J. Spacecr. Rockets* **1**, 403–408 (1964).
5. L. Ting and P. A. Libby, Remarks on the eddy viscosity in compressible mixing flows, *J. Aero. Sci.* **27**, 797–798 (1960).
6. A. Mager, Transformation of the compressible turbulent boundary layer, *J. Aero. Sci.* **25**, 305–311 (1958).
7. W. Forstall Jr. and A. H. Shapiro, Momentum and mass transfer in coaxial jets, *J. Appl. Mech.* **17**, 399–408 (1950).